Interaction-free measurements cannot be perfect

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Abstract. Quantum coherence can be used to infer presence of a detector without triggering it. Here we point out that according to quantum formalism such interaction-free measurements cannot be perfect, i.e. in a single-shot experiment one has strictly positive probability to activate the detector. We provide a quantitative relation between the probability of activation and probability of inconclusive interaction-free measurement in quantum theory and in a more general framework of density cubes. It turns out that the latter does allow for perfect interaction-free measurements. Their absence is therefore a natural postulate expected to hold in all physical theories.

1. Introduction

A sample is seen under the microscope due to photons scattered from it. Similarly, essentially all our knowledge about the physical world comes from probes directly interacting with the objects of interest. Quantum mechanics offers a new possibility for enquiring whether an object is present at a given location — the interaction-free measurement [1]. It is possible by interferometric techniques to prepare a single quantum particle in superposition having one arm in a suspected location of the object and with the measurement scheme which, from time to time, identifies the presence of the object arguably without directly interacting with it [1, 2, 3, 4, 5, 6]. We ask here if interaction-free measurements could be made perfect and provide non-trivial information about the presence of the object even though in each and every run the particle and the object to be detected do not interact directly. Within quantum formalism the answer is negative. If the particle never triggers the detector, the corresponding probability is zero, which implies that the probability amplitude vanishes. There is no way in quantum formalism to learn anything non-trivial about the locations where analysed particles have vanishing probability amplitudes. Indeed it seems plausible that nature also respects the rule of "no perfect interaction-free measurements" and it will be present in any physically meaningful generalisation of the quantum model.

We have therefore identified yet another no-go theorem for quantum mechanics similar to no-cloning [7, 8], no-broadcasting [9, 10] or no-deleting [11]. Their importance comes from pinpointing special features of quantum formalism (and the world) which can then be preserved or relaxed one by one when studying candidate physical theories. In this spirit, here we explore the possibility of perfect interaction-free measurement in a framework of density cubes [12]. It contains quantum mechanics as a special case and allows, among others, for genuine multi-path interference.

Sorkin was the first to note that multi-path quantum interference patterns always reduce to a combination of two-path patterns [13]. He linked this observation to the Born rule: since the number of particles around a given point on the screen is proportional to the square of the sum of wave functions evolved along every path, only products of two wave-functions are responsible for the interference. Experiments were set up to look for genuine multi-path interference and to test the Born rule [14, 15, 16, 17]. Additionally to fundamental interest these experiments also have practical implications as it has been shown that multi-path interference provides advantage over quantum mechanics in the task called "three collision problem" [18], and may be advantageous over quantum algorithms [19], although this is not the case in searching [20]. Up to now essentially all experimental data confirms absence of genuine multi-path interference and consistency of the Born rule. These statements are also supported by additional theoretical research. Namely, models with genuine multi-path interference were shown to be at variance with a number of postulates: purity principles [21, 22, 23], tomography via single-path and double-path experiments [24, 25], and experiment giving a definite (single) outcome [26]. Present work contributes along this line of research by proposing that some models giving rise to multi-path interference allow for perfect interaction-free measurements. As such it is unlikely they would describe natural processes. We note that in the present study it is not only possibility of multi-path interference that empowers perfect interaction-free measurement, but also other elements of formalism. Most notably, there exist transformations which create triple-path coherence that involves a path where the particle is never detected.

We show the existence of perfect interaction-free measurement in a concrete model, so-called density cube model [12]. The basic idea behind this theory is to represent states by higher-rank tensors, density cubes, in direct analogy to quantum mechanical density matrices. In this way more than two classically exclusive possibilities can be coherently coupled giving rise to genuine multi-path interference. The three-path coherence present in the density cube model has indeed been shown to produce irreducible three-path interference patterns and leads to super-quantum violation of the Leggett-Garg inequality [12]. The particular interferometer employed to demonstrate these effects has a feature, also noted by Lee and Selby [18], that the particle is never found in one of the paths inside the interferometer but presence of a detector in that path affects final interference fringes. Therefore, these authors effectively already anticipated possibility of the perfect interaction-free measurement.

The paper is organised as follows. In Sec. 2 we introduce interaction-free measurements and formally define perfect interaction-free measurement. We show that in all theories where processes are assigned probability amplitudes, satisfying natural composition laws, there are no perfect interaction-free measurements and also no genuine multi-path interference. Furthermore, we derive within quantum formalism trade-off relation characterising interaction-free measurements which explicitly shows impossibility of perfect such measurement. We then move to post-quantum models and in Sec. 3 gather all elements of the density cube model necessary for our purposes. Similarly to the quantum case, we derive the trade-off relation within the density cube model, which now opens up a possibility of perfect interaction-free measurement. Section 3.5 provides explicit examples of such measurements. The first example uses three-path interferometer having the property that the particle is never found in the path where we place the detector, but the interaction-free measurement fails 50% of the time (we prove that this cannot be improved using the class of interferometers considered). In next example, we provide N-path interferometer giving rise to perfect interaction-free measurement and vanishing probability of failure in the limit $N \to \infty$. We conclude in section 4.

2. Interaction-free measurements

We begin with the original scheme by Elitzur and Vaidman [1]. The idea is presented and described in Fig. 1. The problem is famously dramatised by considering the presence or absence of a single-particle-sensitive bomb, the tradition we shall also follow.

For a general interferometer (with many paths and arbitrary transformations replacing the beam-splitters in Fig. 1) one always starts by tuning it to destructively interfere in at least one of its output ports. In this way, if the click in one of these ports is observed we conclude that the bomb is present in the setup. This constitutes successful interaction-free measurement and we denote its probability by $P_!$. If the particle emerges in any other output port, we cannot make any definite statement as this happens both in the absence and presence of the bomb. The result is therefore inconclusive and we denote its probability by $P_?$. Finally, if the bomb is present, the probe particle triggers it with probability P_*. Clearly we have exhausted all possibilities and therefore

$$P_* + P_? + P_! = 1. \tag{1}$$

2.1. Perfect interaction-free measurement

We call an interaction-free measurement perfect when statistics of single-shot experiments with the same interferometer shows

$$P_* = 0 \quad \text{and} \quad P_? < 1, \tag{2}$$

i.e. when bomb never explodes and yet from time to time we are certain it was there. The paper ends with an example where both of these probabilities are zero.

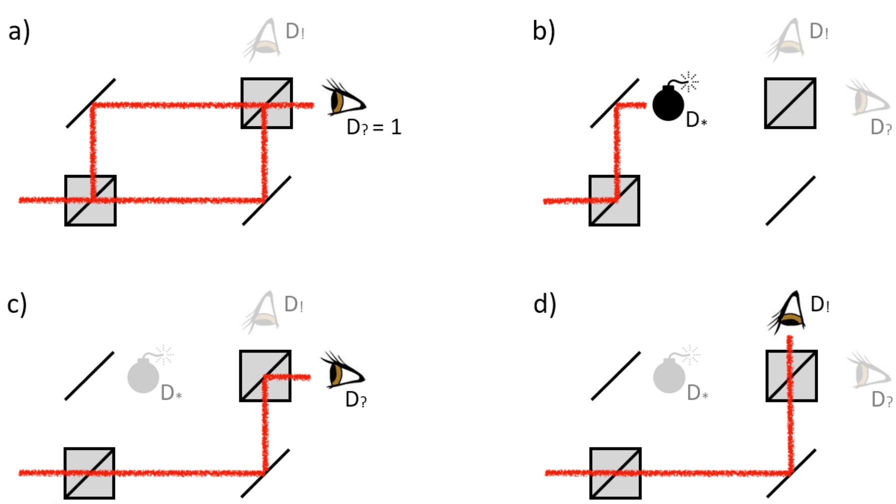

Figure 1. Interaction-free measurement and relevant parameters. Tuned Mach-Zehnder interferometer, as in panel a), is moved to the location where there might be a bomb in the upper arm. The probe particle triggers the bomb (single-particle-sensitive detector D_*) via path b) with probability P_*. The measurement is inconclusive if the particle takes path c) because detector labeled $D_?$ also fires when there is no bomb, see a). The probability of inconclusive result is denoted by $P_?$. Finally, the measurement succeeds if the top detector clicks and this happens with probability $P_!$, see d). The measurement is termed interaction-free because had the particle interacted with the bomb it would trigger it, and it didn't.

2.2. No perfect interaction-free measurements in amplitude models

In quantum mechanics condition $P_* = 0$ implies vanishing probability amplitude for the particle to propagate along the path of the bomb. This means that the first (generalised) beam-splitter never sends the particle to that path and hence it is irrelevant whether one places a bomb there or not, i.e. $P_? = 1$. The same conclusion holds in any theory that assigns probability amplitudes to physical processes and demands that vanishing probability of the process implies vanishing amplitude, e.g. probability is a power of the amplitude. It is intriguing in the present context that many of such models do not give rise to multi-path interference. If the amplitudes are complex numbers (or even pairs of real numbers), their natural composition laws lead to Feynman rules, i.e. probability $\sim$ amplitude^2 [27, 28]. Sorkin's original argument then demonstrates absence of multi-path interference.

Our definition of perfect interaction-free measurement, Eq. (2), demands $P_* = 0$ in every measurement run. In practice one might already be satisfied if P_* is small, so that in M experimental runs the bomb is unlikely to explode. Indeed such a setup exists and makes use of quantum Zeno effect in a looped interferometer [3]. The perfect interaction-free measurements we discuss in Sec. 3.5 can also be looped and provide advantage over the quantum solution in the number of loops one has to execute in order to make $P_?$ small, say equal to ε. The best known quantum solution requires $M \sim 1/\varepsilon$ loops, whereas the post-quantum solution requires only $M = -\log \varepsilon$ and less loops, depending on the number of paths inside the interferometer. Having said this we focus our attention on single-shot experiments and derive now relation between P_* and $P_?$ within quantum formalism.

2.3. Quantum trade-off

Consider a general interferometer as shown in Fig. 2. We present the trade-off between P_* and $P_?$ for arbitrary mixed states but keeping the second transformation unitary, i.e. $\mathcal{T}_2 = \mathcal{U}_2$. Let us denote by ρ the density matrix of the particle inside the interferometer, right after the first transformation. The probability to trigger the bomb is given by

$$P_* = \langle 1| \rho |1\rangle . \tag{3}$$

If the bomb was not triggered state ρ gets updated to $\tilde{\rho}$ satisfying:

$$\tilde{\rho} = \frac{1}{1-P_*}(\mathbb{1} - |1\rangle \langle 1|)\rho(\mathbb{1} - |1\rangle \langle 1|), \tag{4}$$

where $\mathbb{1}$ is the identity operator in the space of density matrices. Accordingly, the particle at the output of the interferometer is described by $\mathcal{U}_2 \tilde{\rho} \mathcal{U}_2^\dagger$. The probability of inconclusive result is given by the chance that now the particle is observed at those output ports $|s\rangle$ in which it might be present if there was no bomb:

$$P_? = (1 - P_*) \sum_s \langle s| \mathcal{U}_2 \tilde{\rho} \mathcal{U}_2^\dagger |s\rangle , \tag{5}$$

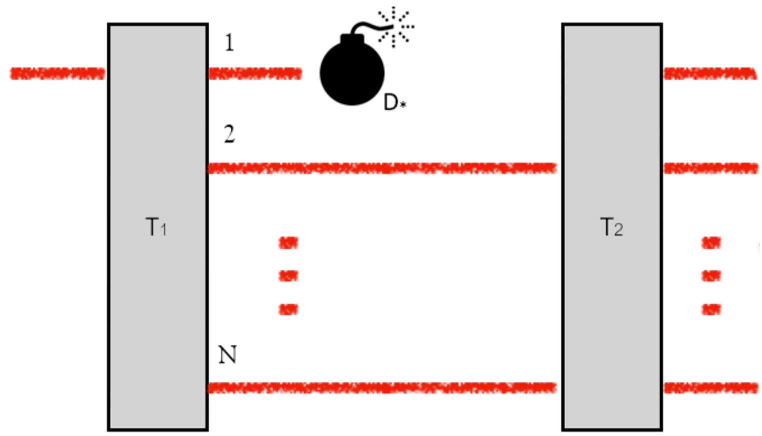

Figure 2. General interferometer used to derive the trade-off relations. The particle is injected into the first path. It then enters the interferometer via transformation $\mathcal{T}_1$ and leaves it via transformation $\mathcal{T}_2$. Inside the interferometer bomb is present in the first path.

where we multiplied by $(1 - P_*)$ to account for the renormalisation in $\tilde{\rho}$. In Appendix A we derive the following trade-off relation:

$$\begin{aligned} P_? &\geq 1 - 2P_* + P_* \langle 1| E(\rho) |1\rangle \\ &\geq (1 - P_*)^2, \end{aligned} \tag{6}$$

where $E(\rho)$ is the projector on the support of ρ, i.e. $E(\rho) = \sum_r |r\rangle \langle r|$ for $\rho = \sum_r p_r |r\rangle \langle r|$. The last inequality in (6) follows from convexity, $\langle 1| E(\rho) |1\rangle \geq \langle 1| \rho |1\rangle$. We now discuss some special cases of this trade-off in order to illustrate tightness of the bound and for future comparison with the cubes model.

First of all, due to convexity, the lower bound is saturated by pure states. In other words pure states are the best for interaction-free measurements. Note also that in quantum formalism by starting with a pure state ρ one always obtains a pure state $\tilde{\rho}$ after the measurement. It turns out that the cube model does not share this property and this could be seen as one of the reasons allowing a perfect interaction-free measurement.

Any density matrix ρ that does not contain coherence to the state $|1\rangle$ is useless for interaction-free measurements. If there is no coherence to state $|1\rangle$, then either (i) one of the eigenvectors of ρ is this state and the others are orthogonal to it or (ii) all the eigenvectors are orthogonal to $|1\rangle$. In the case (i) we find $\langle 1| E(\rho) |1\rangle = 1$ and hence:

$$P_? = 1 - P_*. \tag{7}$$

The trade-off relation becomes an equality due to normalisation of the probabilities. Since both of the probabilities saturate the normalisation bound there is no place for a successful interaction-free measurement. In the case (ii) we note that $P_* = 0$ and hence the lower bound in (6) already shows that $P_? = 1$. This again demonstrates impossibility of perfect interaction-free measurements.

Finally, we note that the trade-off just derived holds for an arbitrary interferometer (with the second transformation being unitary) and is independent of the number of paths. For example, inequality (6) is saturated by taking the discrete Fourier transform as both transformations in the interferometer involving an arbitrary number of paths. This again will be different in the density cube model.

3. Density cubes

In previous section we have derived a quantum trade-off relation between the probability of inconclusive interaction-free measurement, $P_?$, and the probability to trigger the bomb, P_*. This relation could be seen as analogous to the Tsirelson-Landau-Masanes (TLM) bound on the strength of quantum correlations [29, 30, 31]. Both TLM and our inequality provide limits to quantum performance in specific situations. Just like there exist non-trivial post-quantum models which violate the TLM bound [32], there are generalised probabilistic theories where the quantum trade-off between $P_?$ and P_* no longer holds. We shall demonstrate this explicitly using the framework of density cubes [12]. The following three features of this formalism will be shown to lead to a perfect interaction-free measurement: (i) triple-path coherence, (ii) existence of transformation which produces coherence involving a path where the particle is never detected and (iii) state update in a measurement.

The model of density cubes follows closely the usual mathematical formulation of quantum theory. The main difference is that instead of quantum density matrix, one assigns a rank-3 tensor (density cube) to a given physical configuration. Other elements of the model are chosen such that a natural probability rule holds (immediate extension of the Born rule for mixed states) and the model is self-consistent. The set of allowed density cubes and their transformations are not yet fully characterised at present and it is not our aim to characterise them in this paper. We will rather focus on specific density cubes and transformations, that will be shown to be consistent and will produce perfect interaction-free measurement. In the following we review the details of the model.

3.1. Probability

The basic entity of the model is a density cube C: a rank-3 tensor with complex elements $C_{jkl} \in \mathbb{C}$. The density cubes are assumed to be Hermitian in a sense that exchanging any two indices produces a complex conjugated element, e.g.:

$$C_{jkl} = C^*_{kjl}. \tag{8}$$

Hermitian cubes form a real vector space with inner product

$$(M, C) = \sum_{j,k,l=1}^{N} M^*_{jkl} C_{jkl}, \tag{9}$$

where each index of the tensor runs through values $1, \ldots, N$. Therefore, one naturally defines the probability to observe outcome corresponding to cube M in a measurement on a physical object described by cube C by the above inner product. This is in close analogy to the Born rule in quantum mechanics which in the same situation assigns probability $\mathrm{Tr}(MC) = \sum_{j,k=1}^{N} M_{jk}^{*} C_{jk}$, with M and C being density matrices. In this way the model of density cubes extends self-duality between states and measurements present in quantum mechanics [33, 34].

3.2. States

We shall consider two types of density cubes: the quantum cubes which represent quantum states in the density cube model and non-quantum cubes (with triple-path coherence) which extend the quantum set. The former are constructed from quantum states and are in one-to-one relation with the quantum states. While non-quantum cubes are also constructed starting from a quantum state, one can choose various combinations for the triple-path coherence terms to construct several distinct non-quantum cubes corresponding to a given quantum state.

3.2.1. Quantum cubes Consider the following mapping between a density matrix ρ and a cube C^{Q}:

$$\begin{aligned} C_{jjj}^{Q} &= \rho_{jj}, \\ C_{jjk}^{Q} &= \sqrt{\tfrac{2}{3}}\,\mathrm{Re}(\rho_{jk}), \quad \text{for} \quad j<k, \\ C_{jkk}^{Q} &= \sqrt{\tfrac{2}{3}}\,\mathrm{Im}(\rho_{jk}), \quad \text{for} \quad j<k, \\ C_{jkl}^{Q} &= 0, \quad \text{for} \quad j \neq k \neq l. \end{aligned} \tag{10}$$

Note that all the terms C_{jkl}^{Q} where the three indices are different are set to zero, meaning that these cubes do not admit any three-path coherence. The remaining elements can be computed using the Hermiticity rule. This mapping preserves the inner product between the states and hence quantum mechanics and density cube model with this set of cubes are physically equivalent.

3.2.2. Non-quantum cubes We now extend the set of quantum cubes and allow for non-trivial triple-path coherence by mapping every quantum state ρ to the following family of cubes:

$$\begin{aligned} C_{jjj} &= \frac{1}{N-1}(1-\rho_{jj}), \\ C_{jjk} &= \sqrt{\frac{2}{3}}\frac{1}{N-1}\,\mathrm{Re}(\rho_{jk}), \quad \text{for} \quad j<k, \\ C_{jkk} &= \sqrt{\frac{2}{3}}\frac{1}{N-1}\,\mathrm{Im}(\rho_{jk}), \quad \text{for} \quad j<k, \\ C_{1jk}(\gamma) &= \sqrt{\frac{1}{3}}\frac{1}{N-1}\,\omega^{f(\gamma,j,k)}, \quad \text{for} \quad 1<j<k, \end{aligned} \tag{11}$$

where $\omega = \exp(-i2\pi/N)$ is the Nth complex root of unity and $f(\gamma, j, k) = \{1, \ldots, N\}$. Parameter $\gamma = 1, \ldots, N$ enumerates different cubes that can be constructed from a given quantum state. Again, the remaining elements can be completed using the Hermiticity rule. We provide explicit examples of interesting non-quantum cubes in section 3.5

and Appendix C. Note that for simplicity we choose to place the bomb in the first path of the interferometer and therefore consider cubes where the three-path coherence involves only state 1 (the first path) and two other states. All the terms C_{jkl}, with three different indices, each of which is strictly greater than 1, are set to zero.

It is convenient to distinguish a *quantum part* of the non-quantum cube. The quantum part $Q(C)$ is obtained by applying isomorphism (10) to cube C assuming its three-path coherences vanish. One immediately finds:

$$Q(C) = \frac{1}{N-1}(\mathbb{1} - \rho), \tag{12}$$

where $\mathbb{1}$ denotes $N \times N$ identity matrix. The probability rule now splits into the quantum part and the part depending only on the three-path coherences:

$$(M, C) = \mathrm{Tr}[Q(M)Q(C)] + \sum_{j \neq k \neq l} M^*_{jkl}(\gamma')C_{jkl}(\gamma), \tag{13}$$

where γ' and γ specify the cubes M and C according to Eq. (11). It is now clear that non-quantum cubes have non-negative overlap (as it should be for a probability) with all the quantum cubes, because the latter do not give rise to three-path coherence. We still have to ensure that the inner products between all the non-quantum cubes are non-negative and bounded by one. The set of non-quantum cubes of interest here will be shown to be finite and satisfying this requirement.

3.3. Measurement

We shall only be interested in enquiring about particle's path at various stages of the evolution. Furthermore, we will focus on checking whether the particle is in the first path or not. Clearly this measurement is allowed in quantum mechanics and we choose standard basis as the basis of which-path measurement. The corresponding quantum cube looks as follows in the case of triple-path experiment:

$$M_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{14}$$

where the three 3×3 matrices describing the cube have elements C_{1jk}, C_{2jk} and C_{3jk}, respectively. The probability that a particle described by cube C is found in the first path is (M_1, C).

It is essential to the interaction-free measurement to describe the state of the particle after it has *not* been found in a particular path. Here the model of density cubes also follows quantum mechanics and it is assumed that the cube describing the system changes as a result of measurement. If the particle is found in the nth path its state gets updated $C \to M_n$, where M_n is the quantum cube corresponding to a particle propagating along the nth path. If the particle is not found in the nth path the model follows generalised Lüder's state update rule: erases from the cube all elements C_{jkl} with $j, k, l = n$, and renormalises the remaining elements. Following our three-path example, if the particle is not found in the first path its generic cube C gets updated to:

$$\tilde{C} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{C}_{222} & \tilde{C}_{223} \\ 0 & \tilde{C}_{232} & \tilde{C}_{233} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{C}_{322} & \tilde{C}_{323} \\ 0 & \tilde{C}_{332} & \tilde{C}_{333} \end{pmatrix} \right\}, \tag{15}$$

where

$$\tilde{C}_{jkl} = \frac{1}{1 - C_{111}} C_{jkl}, \tag{16}$$

is the cube element renormalised by the probability that the particle is not in the first path.

At this stage it has to be ensured that all post-measurement cubes are allowed within the model. This is immediately clear if one begins with a quantum cube. For the non-quantum cubes we note that we only consider those cubes which have three-path coherence to the first path and we only enquire whether the particle is in the first path or not. If the measurement does not find the particle in the first path all these coherences are updated to zero and accordingly the post-measurement cube is a quantum one.

3.4. Cubes trade-off

We are now ready to present the trade-off relation between P_* and $P_?$ for a general interferometer in Fig. 2. Our trade-off relation holds for transformation $\mathcal{T}_2$ that preserves the inner product, while having an additional assumption on the structure of cube C describing the particle inside the interferometer right after $\mathcal{T}_1$. We assume that after the particle has propagated through the whole interferometer in the case of no bomb, the cube at the output does not have any two-path and three-path coherence:

$$\mathcal{T}_2(C) = \sum_s p_s M_s. \tag{17}$$

We ensure this is always fulfilled in our examples. Similarly to the quantum case, the probabilities entering the trade-off are defined as follows:

$$\begin{aligned} P_* &= (M_1, C), \\ P_? &= (1 - P_*) \sum_s (M_s, \mathcal{T}_2(\tilde{C})), \end{aligned} \tag{18}$$

where the cube $\tilde{C}$ represents the particle inside the interferometer after the measurement in the first path has not found the particle there, see Eq. (15). In Appendix B we derive the trade-off relation within the cubes model:

$$P_? \geq \frac{(1 - P_*)^2}{N - 1}. \tag{19}$$

It is illustrated in Fig. 3. One recognises that for $N = 2$ this relation reduces to the one derived in quantum mechanics. For two-path interferometers this is not surprising as in this case the density cube model reduces to standard quantum formalism [12]. For higher number of paths, this relation emphasises that independence of the number of paths is a special quantum feature.

Relation (19) opens up a possibility of perfect interaction-free measurements. Indeed for all $N \geq 3$ one finds that the right-hand side is strictly less than 1 even if $P_* = 0$. Furthermore, both probabilities P_* and $P_?$ can in principle be brought to zero in the limit $N \to \infty$. In the next section we provide explicit examples of perfect interaction-free measurements which achieve the lower bound set by the trade-off relation (19).

3.5. Examples of perfect interaction-free measurements

We present in detail the workings of the perfect interaction-free measurement in the case of three-path interferometer with emphasis on the features departing from quantum formalism. Subsequent section provides generalisation to N paths. We discuss the main idea here and refer to Appendix D for the details.

3.5.1. Three paths Consider the setup described in Fig. 4. The transformation $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T} \circ \mathcal{D}$ is chosen to consist of (quantum mechanical) complete dephasing of two-path coherences, $\mathcal{D}$, followed by the transformation $\mathcal{T}$ defined in Eq. (16) of Ref. [12], which we will here review for completeness. The reason behind this composition of

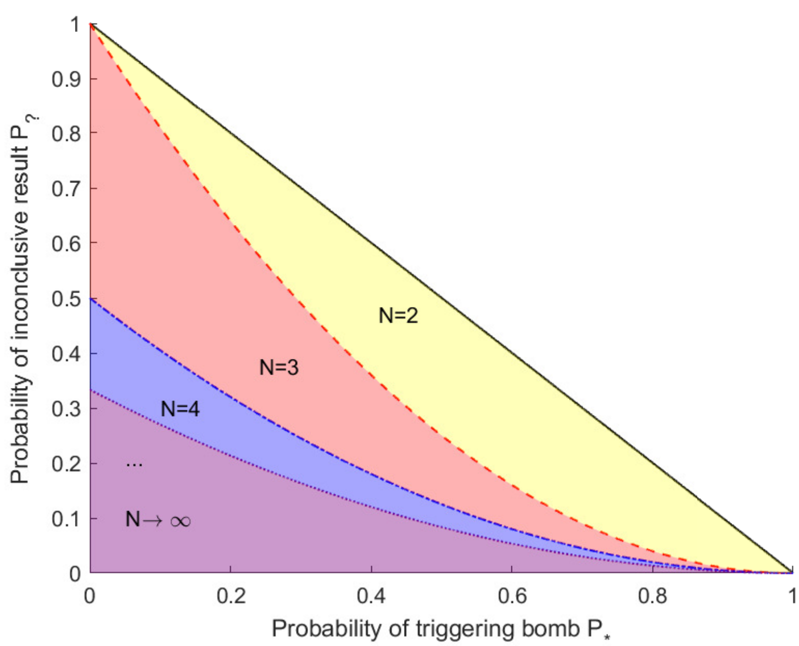

Figure 3. Trade-off between the probability to trigger the bomb, P_*, and the probability of inconclusive result, $P_?$, within the cubes model and quantum mechanics. The straight line illustrates the trivial bound $P_* + P_? = 1$. All other region borders give lower bounds on the value of $P_?$ as a function of P_*. The available region for any N (number of paths inside the interferometer) contains also the regions for all lower values of N. The quantum mechanical trade-off coincides with the case of $N = 2$. Perfect interaction-free measurements occur if the allowed values on the vertical axis are less than 1.

operations is that $\mathcal{T}$ is only defined on a subset of cubes and it might be that it is impossible to extend it consistently to the whole set of cubes. The role of the dephasing is then to bring an arbitrary cube to the subset on which $\mathcal{T}$ is known to act consistently. The dephasing operation is defined to remove completely all two-path coherences in a cube and leave unaffected the diagonal elements C_{nnn} and triple-path coherences C_{jkl} with all indices different. Since this operation acts only on the quantum part of the cube it produces allowed cubes. Transformation $\mathcal{T}$ has matrix representation

$$\mathcal{T} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega^* & \omega \\ 1 & 1 & 0 & \omega & \omega^* \\ 1 & \omega & \omega^* & 1 & 0 \\ 1 & \omega^* & \omega & 0 & 1 \end{pmatrix}, \tag{20}$$

when written in the following sub-basis of Hermitian cubes:

$$B_1 = M_1, \qquad B_2 = M_2, \qquad B_3 = M_3,$$

$$B_4 = \frac{1}{\sqrt{3}}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\},$$

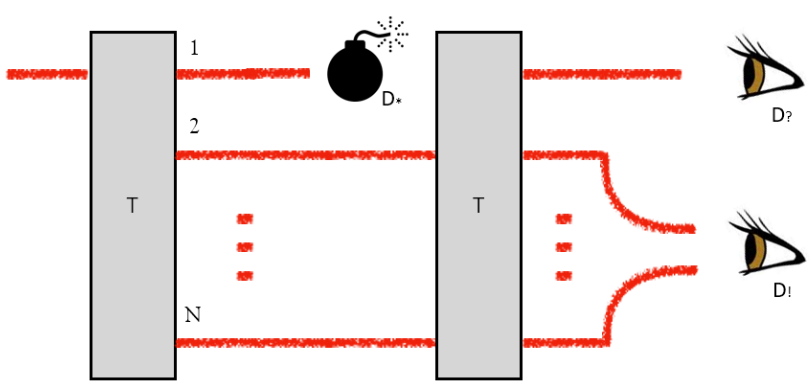

Figure 4. Perfect interaction-free measurement within the density cube model. Both transformations are the same $\mathcal{T}_1 = \mathcal{T}_2$ and they have the property that $\mathcal{T}_1^2 = \mathbb{1}$. Therefore, if there is no bomb the particle which enters through the first path always leaves the interferometer along the first path. The cube describing the particle inside the interferometer has only triple-path coherences to the first path and yet vanishing element C_{111}. Therefore the particle is never found along the first path inside the interferometer and the bomb never detonates, $P_* = 0$. But the presence of the bomb removes the triple-path coherences from the cube, and in this case the second transformation evolves the particle to the first output port only with probability $P_? = \frac{1}{2}$. See main text for the details.

$$B_5 = \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{21}$$

That is, given an arbitrary cube C in this subspace, the transformation $\mathcal{T}$ acts upon it via ordinary matrix multiplication on the vector representation of C, i.e. a five-dimensional column vector with jth component given by (C, B_j). As already alluded to, this subspace consists of cubes which have no two-path coherences, but solely three-path coherences and the diagonal terms. It is now straightforward to verify that $\mathcal{T}$ is an involution in the considered subspace, i.e. $\mathcal{T}^2 = \mathbb{1}$. Accordingly, if the particle enters the interferometer though the first path, it is always found in the first output port of the setup. This adheres to our assumption (17) as the output cube is simply M_1.

Transformation $\mathcal{T}$ is different from arbitrary unitary transformation as it produces triple-path coherence inside the interferometer. The particle injected into the first path is described by the cube M_1, which in the considered subspace corresponds to the first vector in the computational basis, and one can verify that the corresponding cube after application of $\mathcal{T}$ is $\mathcal{T}(M_1) = \frac{1}{2}(B_2 + B_3 + B_4 + B_5)$, which is also given by:

$$C = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \right\}. \tag{22}$$

Note that it is a pure cube, i.e. $(C, C) = 1$, and it contains solely three-path coherences and elements C_{222} and C_{333}. The essential feature we are utilising for perfect interaction-free measurement is the presence of these coherences even though the probability to find the particle in the first path vanishes

$$P_* = (M_1, C) = 0. \tag{23}$$

A similar statement for quantum states does not hold. If the probability to locate a quantum particle in the first path vanishes, all coherences to this path must vanish, as otherwise the corresponding density matrix has negative eigenvalues.

If the bomb is present inside the interferometer but it is not triggered the state update rule dictates to erase all elements C_{jkl} with any of $j, k, l = 1$. We obtain the following cube:

$$\tilde{C} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \right\}. \tag{24}$$

It contains no coherences whatsoever and it is mixed, i.e. $(\tilde{C}, \tilde{C}) = \frac{1}{2}$. We started with a pure cube and post-selected a mixed one. This is also not allowed within quantum formalism, where any pure state $|\psi\rangle = \sum_{n=1}^{N} \alpha_n |n\rangle$ gets updated to another pure state $|\tilde{\psi}\rangle = \sum_{n=2}^{N} \tilde{\alpha}_n |n\rangle$, with $\tilde{\alpha}_n = \alpha_n / \sqrt{1 - |\alpha_1|^2}$.

Finally, we evolve $\tilde{C} = \frac{1}{2}M_2 + \frac{1}{2}M_3$ through the second transformation and find that $\mathcal{T}(\tilde{C})$ is given by (dephasing has no effect here):

$$\left\{ \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{4\sqrt{3}} \\ 0 & -\frac{1}{4\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{4\sqrt{3}} \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{4\sqrt{3}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{4\sqrt{3}} & 0 \\ -\frac{1}{4\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \right\}.$$

The probability of inconclusive result is given by the probability that the particle is found in the first path, as it was always there in the absence of the bomb, and therefore we find:

$$P_? = (M_1, \mathcal{T}(\tilde{C})) = \frac{1}{2}. \tag{25}$$

This probability saturates the lower bound derived in Eq. (19) for $N = 3$ and hence the setup discussed is optimal.

3.5.2. More than three paths We now generalise above scheme to more than three paths and show that the density cube model allows for perfect interaction-free measurement which in every run provides complete information about the presence of the bomb. However, this only holds in the limit $N \to \infty$.

We shall now construct a set of N pure orthonormal cubes, $C^{(n)}$, which will then be used to provide transformation $\mathcal{T}$ of the optimal interferometer, i.e. giving rise to the minimal probability of inconclusive result while keeping $P_* = 0$. We set the modulus of all the three-path coherences within each cube $C^{(n)}$ to be the same and choose its non-zero elements as follows:

$$\begin{aligned} C^{(n)}_{jjj} &= \frac{1}{N-1}, && \text{for } j \neq n, \\ C^{(n)}_{1jk} &= \sqrt{\frac{1}{3}}\frac{1}{N-1}\, x^{(n)}_{jk}, && \text{for } 1 < j < k. \end{aligned} \tag{26}$$

The other non-zero three-path coherences can be found from the Hermiticity rule. In this way cube $C^{(n)}$ is represented by a set of phases $x^{(n)}_{jk}$. We arrange the independent phases, i.e. the ones having $j < k$, into a vector $\vec{x}_n$. The orthonormality conditions between the cubes are now expressed in the following equations

$$\begin{aligned} (C^{(n)}, C^{(n)}) = 1 &\iff |(\vec{x}_n)_j| = 1 \text{ for all } n, j, \\ (C^{(m)}, C^{(n)}) = 0 &\iff (\vec{x}_m, \vec{x}_n) + (\vec{x}_n, \vec{x}_m) = 2 - N, \text{ for all } m \neq n. \end{aligned} \tag{27}$$

where $(\vec{x}_n)_j$ is the jth component of vector $\vec{x}_n$. Equations (27) are solved in Appendix C. Let us write the solution in form of a matrix

$$X = (\vec{x}_1 \dots \vec{x}_N), \tag{28}$$

having vectors $\vec{x}_n$ as columns. We now show how to use it to construct the "cube multiport" transformation $\mathcal{T}$.

We assume the two transformations in the setup are the same and that $\mathcal{T}$ is defined solely on the subspace of Hermitian cubes which do not have any two-path coherences.

The cubes forming the basis set for this subspace are as follows:

$$B_{jkl}^{(n)} = \delta_{jn}\delta_{kn}\delta_{ln}, \quad \text{for } n = 1, \dots, N \tag{29}$$

$$B_{jkl}^{(vw)} = \frac{1}{\sqrt{3}}\left(\delta_{j1}\delta_{kv}\delta_{lw} + \delta_{jw}\delta_{k1}\delta_{lv} + \delta_{jv}\delta_{kw}\delta_{l1}\right), \quad \text{for } 1 < v < w \leq N,$$

$$B_{jkl}^{(wv)} = \frac{1}{\sqrt{3}}\left(\delta_{j1}\delta_{kw}\delta_{lv} + \delta_{jv}\delta_{k1}\delta_{lw} + \delta_{jw}\delta_{kv}\delta_{l1}\right), \quad \text{for } 1 < v < w \leq N.$$

One recognises that the cubes in the first line are just the M_n cubes describing the particle propagating along the nth path. The cubes in the second line describe independent three-path coherences, and the cubes in the third line their complex conjugations. Altogether there are $d = N + (N-1)(N-2)$ cubes in this sub-basis and hence transformation $\mathcal{T}$ is represented by $d \times d$ matrix, which we then divide into blocks:

$$\mathcal{T} = \left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right), \tag{30}$$

A being a square $N \times N$ matrix, D being a square matrix with dimension $(N-1)(N-2) \times (N-1)(N-2)$, and B and C being rectangular. By imposing that $\mathcal{T}(M_n) = C^{(n)}$ matrices A and B are fixed to

$$A = \frac{1}{N-1}\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} X \\ X^* \end{pmatrix}. \tag{31}$$

By further requiring involution $\mathcal{T}^2 = \mathbb{1}$ and Hermiticity $\mathcal{T} = \mathcal{T}^\dagger$ one finds that

$$C = B^\dagger, \qquad D = \sqrt{\mathbb{1} - BB^\dagger}. \tag{32}$$

We show in Appendix D that $\mathbb{1} - BB^\dagger$ is a positive matrix, which concludes our construction of $\mathcal{T}$. It turns out that this is not the only way to construct the cube multiport transformation and Appendix D provides other examples. All of them transform the quantum cubes M_n to the non-quantum cubes $C^{(n)}$. Note that in the considered subspace M_n are the only pure quantum cubes and one verifies that $C^{(n)}$ are the only pure non-quantum cubes allowed. In this way $\mathcal{T}$ is shown to act consistently, i.e. map cubes allowed within the model to other allowed cubes.

4. Conclusions

We proposed a theory-independent definition of perfect interaction-free measurement. It turns out that quantum mechanics does not allow this possibility but it can be realised within the framework of density cubes [12]. Notably, this framework allows transformations that prepare triple-path coherence involving a path where the probability of detecting the particle is strictly zero. Nevertheless, this coherence can be destroyed if the bomb (detector) is in the setup, leading to a distinguishable outcome in a suitable one-shot interference experiment. (We emphasise that here we study single-shot experiments in contrast to quantum Zeno effect which allows for perfect interaction-free measurements in the limit of infinitely many uses of the interferometer [3]). We therefore propose that such operations should not be present in a physical theory as they effectively allow deduction of the presence of an object in a particular location without ever detecting a particle in that location. This might be considered as a natural postulate about physical theories or one might also try to identify other more basic postulates which imply impossibility of perfect interaction-free measurements.

In this context we note that perfect interaction-free measurements are consistent with the no-signalling principle (no super-luminar communication). In the density cube model it is the triple-path coherence that is being destroyed by the presence of the detector inside the interferometer. The statistics of any observable measured on the remaining paths is the same independently of whether the detector is in the setup or not. Hence the information about its presence can only be acquired after recombining the paths together, which can be done at most with the speed of light. The situation resembles that of the stronger than quantum correlations satisfying the principle of no-signalling [32]. They are considered "too strong" as they trivialise communication complexity [35, 36] or random access coding [37], and they are at variance with many natural postulates [37, 38, 39]. Similarly, we consider identifying presence of a detector without ever triggering it, i.e. a perfect interaction-free measurement, as too powerful to be realised in nature. Exactly which physical principles forbid such measurements is, of course, an interesting question.

Finally, we wish to comment briefly on experimental tests of genuine multi-path interference. They are often described as simultaneously testing validity of Born's rule. Indeed, as we also pointed out here, this is the case for a broad class of models which assign probability amplitudes to physical processes and these amplitudes satisfy natural composition laws [27, 28]. Other models, however, are possible as exemplified by the density cube framework. Within this framework the probability rule is essentially the same as the Born rule in quantum mechanics [its version for mixed states, see Eq. (9)]. Therefore, in general, tests of multi-path interference should be distinguished from validity tests of Born's rule.

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Appendix A. Proof of the quantum trade-off

Let us denote the eigenstates of a density matrix ρ describing the particle inside the interferometer right after the first transformation as $|r\rangle$, i.e. $\rho = \sum_r p_r |r\rangle\langle r|$. We also write $\mathcal{U}_2 |r\rangle = |\phi_r\rangle$. From the definition of the probability of inconclusive result:

$$\frac{P_?}{1-P_*} = \mathrm{Tr}\left(\sum_s |s\rangle\langle s| \mathcal{U}_2\, \tilde{\rho}\, \mathcal{U}_2^\dagger\right), \tag{A.1}$$

where the sum is over the paths $|s\rangle$ at the output of the interferometer where the particle could be found if there was no bomb, i.e. if $\mathcal{U}_2\, \rho\, \mathcal{U}_2^\dagger = \sum_r p_r |\phi_r\rangle\langle\phi_r|$ is the state at the output. Therefore, states $|s\rangle$ span a subspace that contains the eigenstates $|\phi_r\rangle$ and we conclude

$$\sum_s |s\rangle\langle s| = \sum_r |\phi_r\rangle\langle\phi_r| + \sum_\mu |\mu\rangle\langle\mu|, \tag{A.2}$$

where $|\mu\rangle$'s complement the subspace spanned by the paths. Since $\langle\mu| \mathcal{U}_2\, \tilde{\rho}\, \mathcal{U}_2^\dagger |\mu\rangle \geq 0$, Eq. (A.1) admits the lower bound:

$$\frac{P_?}{1-P_*} \geq \mathrm{Tr}\left(\sum_r |\phi_r\rangle\langle\phi_r| \mathcal{U}_2\, \tilde{\rho}\, \mathcal{U}_2^\dagger\right) = \mathrm{Tr}\left(\sum_r |r\rangle\langle r|\, \tilde{\rho}\right). \tag{A.3}$$

Using the definition of $\tilde{\rho}$ in terms of ρ given in Eq. (4) of the main text we obtain

$$P_? \geq 1 - 2P_* + P_* \langle 1| E(\rho) |1\rangle, \tag{A.4}$$

with $E(\rho) = \sum_r |r\rangle\langle r|$.

Appendix B. Proof of the cubes trade-off

Let us first recall our assumption about cube C describing the particle inside the interferometer [Eq. (17) of the main text]:

$$\mathcal{T}_2(C) = \sum_s p_s M_s. \tag{B.1}$$

The following steps form the first part of the derivation:

$$\begin{aligned} \frac{P_?}{1-P_*} &= \sum_s (M_s, \mathcal{T}_2(\tilde{C})) \\ &\geq \sum_s (p_s M_s, \mathcal{T}_2(\tilde{C})) = (\mathcal{T}_2(C), \mathcal{T}_2(\tilde{C})) = (C, \tilde{C}). \end{aligned} \tag{B.2}$$

The first line is the definition of the probability of inconclusive result, the inequality follows from convexity, then we used (B.1) and finally the fact that $\mathcal{T}_2$ preserves the inner product. In the second part we shall find the minimum of the right-hand side. Using the expression for the elements of $\tilde{C}$ in terms of elements of C we find:

$$(C, \tilde{C}) = \frac{1}{1-P_*} \sum_{j,k,l=2}^{N} |C_{jkl}|^2, \tag{B.3}$$

where $C_{222} + \ldots + C_{NNN} = 1 - P_*$. Since all of the summands are non-negative we get the lower bound by setting all the off-diagonal terms to zero. It is then easy to verify that the minimum is achieved for even distribution of the probability:

$$C_{nnn} = \frac{1 - P_*}{N - 1} \quad \text{for } n = 2, \ldots, N. \tag{B.4}$$

Using this lower bound in (B.2) we obtain:

$$P_? \geq \frac{(1 - P_*)^2}{N - 1}. \tag{B.5}$$

Appendix C. Solution to the orthonormality equations for optimal cubes

The solution is divided into two parts: N even and N odd.

Appendix C.1. N even

Let M be a $(N-1) \times N$ matrix formed from the discrete $N \times N$ Fourier transform by deleting the first row:

$$M = \begin{pmatrix} 1 & \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(N-1)} \\ 1 & \omega^{2\cdot 1} & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-1)\cdot 1} & \omega^{(N-1)\cdot 2} & \cdots & \omega^{(N-1)\cdot(N-1)} \end{pmatrix}, \tag{C.1}$$

where $\omega = \exp(i2\pi/N)$. The crucial property we shall use is expressed in the following multiplication:

$$M^\dagger M = \begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-1 \end{pmatrix}. \tag{C.2}$$

Hence, the columns of matrix M form vectors with fixed overlap equal to -1, for any pair of distinct vectors. Let us now form matrix X by stacking $(N-2)/2$ matrices M vertically:

$$X = \begin{pmatrix} M \\ M \\ \vdots \\ M \end{pmatrix}. \tag{C.3}$$

Note that matrix X has N columns and $(N-1)(N-2)/2$ rows. We therefore define vectors $\vec{x}_n$ as columns of X:

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_N \end{pmatrix}. \tag{C.4}$$

Indeed, every component of each $\vec{x}_n$ has unit modulus and appropriate overlap:

$$\begin{aligned} &(\vec{x}_m, \vec{x}_n) + (\vec{x}_n, \vec{x}_m) = 2(\vec{x}_m, \vec{x}_n) = 2(m\text{th row of } X^\dagger)(n\text{th row of } X) \\ &= (N-2)(m\text{th row of } M^\dagger)(n\text{th row of } M) = 2 - N. \end{aligned} \tag{C.5}$$

Appendix C.2. N odd

Now we construct matrix M, having dimensions $\frac{N-1}{2} \times N$, by deleting the first row of the $N \times N$ Fourier transform matrix and taking only $\frac{N-1}{2}$ top rows left:

$$M = \begin{pmatrix} 1 & \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot N} \\ 1 & \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{\frac{N-1}{2}\cdot 1} & \omega^{\frac{N-1}{2}\cdot 2} & \cdots & \omega^{\frac{N-1}{2}\cdot N} \end{pmatrix}. \tag{C.6}$$

This time we have:

$$M^\dagger M = \begin{pmatrix} \frac{N-1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & \frac{N-1}{2} & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & \frac{N-1}{2} \end{pmatrix} + i\,(\text{imaginary part}). \tag{C.7}$$

We form matrix X by stacking $N-2$ matrices M vertically and define vectors $\vec{x}_n$ as columns of X as before. Indeed the overlap between distinct vectors reads:

$$(\vec{x}_m, \vec{x}_n) + (\vec{x}_n, \vec{x}_m) = 2\text{Re}\left[(\vec{x}_m, \vec{x}_n)\right] \tag{C.8}$$

$$= 2\text{Re}\left[(m\text{th row of } X^\dagger)(n\text{th row of } X)\right] \tag{C.9}$$

$$= 2(N-2)\text{Re}\left[(m\text{th row of } M^\dagger)(n\text{th row of } M)\right] = 2 - N. \tag{C.10}$$

Appendix C.3. Example of resulting cubes for N = 4

The following four cubes are obtained for the four-path interferometer. Each cube below has to be multiplied by $\frac{1}{3\sqrt{3}}$.

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & \sqrt{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix} \right\}, \tag{C.11}$$

$$\left\{ \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -i & -1 \\ 0 & i & 0 & i \\ 0 & -1 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 & i \\ i & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -i & 0 \\ -1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & \sqrt{3} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & -i & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i & -1 \\ 0 & \sqrt{3} & 0 & 0 \\ i & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & i & 0 \\ -1 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Appendix D. Consistency of the transformation

Appendix D.1. Positivity of matrix D

We shall find the eigenvalues of $\mathbb{1} - BB^\dagger$ explicitly. Since $B = (X\ X^*)^t$, where t denotes transpose, the construction in Appendix C for even N gives

$$BB^\dagger = \frac{1}{(N-1)^2}\left(\begin{array}{ccc|ccc} MM^\dagger & \cdots & MM^\dagger & M^*M^\dagger & \cdots & M^*M^\dagger \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ MM^\dagger & \cdots & MM^\dagger & M^*M^\dagger & \cdots & M^*M^\dagger \\ \hline MM^t & \cdots & MM^t & M^*M^t & \cdots & M^*M^t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ MM^t & \cdots & MM^t & M^*M^t & \cdots & M^*M^t \end{array}\right), \quad \text{(D.1)}$$

Note that $MM^\dagger = N\,\mathbb{1}$ inherited from the unitarity of the Fourier matrix. Similarly $M^*M^t = (M^\dagger)^t M^t = (MM^\dagger)^t = N\,\mathbb{1}$. Direct computation shows $MM^t = M^*M^\dagger = N\,\mathbb{1}^A$, where $\mathbb{1}^A$ denotes an anti-diagonal matrix with all non-zero elements being 1. Therefore, matrix $(N-1)^2 BB^\dagger$ has the following form for even N:

$$\left(\begin{array}{ccc:c:ccc|ccc:c:ccc}
1 & & & & 1 & & & & & 1 & & & & 1 \\
 & \ddots & & \cdots & & \ddots & & & \iddots & & \cdots & & \iddots & \\
 & & 1 & & & & 1 & 1 & & & & 1 & & \\ \hdashline
 & \vdots & & & & \vdots & & & \vdots & & & & \vdots & \\ \hdashline
1 & & & & 1 & & & & & 1 & & & & 1 \\
 & \ddots & & \cdots & & \ddots & & & \iddots & & \cdots & & \iddots & \\
 & & 1 & & & & 1 & 1 & & & & 1 & & \\ \hline
 & & 1 & & & & 1 & 1 & & & & 1 & & \\
 & \iddots & & \cdots & & \iddots & & & \ddots & & \cdots & & \ddots & \\
1 & & & & 1 & & & & & 1 & & & & 1 \\ \hdashline
 & \vdots & & & & \vdots & & & \vdots & & & & \vdots & \\ \hdashline
 & & 1 & & & & 1 & 1 & & & & 1 & & \\
 & \iddots & & \cdots & & \iddots & & & \ddots & & \cdots & & \ddots & \\
1 & & & & 1 & & & & & 1 & & & & 1
\end{array}\right). \quad \text{(D.2)}$$

Similarly one finds for odd N:

$$\left(\begin{array}{c|c}
\begin{array}{ccc:c:ccc}
1 & & & & 1 & & \\
& \ddots & & \cdots & & \ddots & \\
& & 1 & & & & 1 \\ \hdashline
& \vdots & & & & \vdots & \\ \hdashline
1 & & & & 1 & & \\
& \ddots & & \cdots & & \ddots & \\
& & 1 & & & & 1
\end{array} & 0 \\ \hline
0 & \begin{array}{ccc:c:ccc}
1 & & & & 1 & & \\
& \ddots & & \cdots & & \ddots & \\
& & 1 & & & & 1 \\ \hdashline
& \vdots & & & & \vdots & \\ \hdashline
1 & & & & 1 & & \\
& \ddots & & \cdots & & \ddots & \\
& & 1 & & & & 1
\end{array}
\end{array}\right). \tag{D.3}$$

In both cases of N being even or odd the eigenvalues of $BB^\dagger$ are either 0 or $N(N-2)/(N-1)^2$. Hence the eigenvalues of D are either 1 or $1/(N-1)^2$.

Appendix D.2. Exemplary transformation for $N = 4$

We shall only present the D matrices. Following the method above one finds

$$D = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{D.4}$$

This is not the only solution given the constraints of $\mathcal{T}^2 = \mathbb{1}$ and $\mathcal{T} = \mathcal{T}^\dagger$. The following two matrices were obtained by other means:

$$D_2 = \frac{1}{3}\begin{pmatrix} 1 & -i & i & -i & i & 0 \\ i & 1 & 1 & -1 & 0 & -i \\ -i & 1 & 1 & 0 & -1 & i \\ i & -1 & 0 & 1 & 1 & -i \\ -i & 0 & -1 & 1 & 1 & i \\ 0 & i & -i & i & -i & 1 \end{pmatrix}, \tag{D.5}$$

$$D_3 = \frac{1}{3}\begin{pmatrix} 1 & -1 & -i & i & 1 & 0 \\ -1 & 1 & -i & i & 0 & 1 \\ i & i & 1 & 0 & -i & -i \\ -i & -i & 0 & 1 & i & i \\ 1 & 0 & i & -i & 1 & -1 \\ 0 & 1 & i & -i & -1 & 1 \end{pmatrix}. \tag{D.6}$$

If we use D_3 as an example, then $\mathcal{T}$ has the following elements:

$$\mathcal{T} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & i & -1 & -i & -i & -1 & i \\ 1 & 1 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 0 & -i & -1 & i & i & -1 & -i \\ 1 & -i & -1 & i & 1 & -1 & -i & i & 1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & -i & i & 0 & 1 \\ 1 & i & -1 & -i & i & i & 1 & 0 & -i & -i \\ 1 & i & -1 & -i & -i & -i & 0 & 1 & i & i \\ 1 & -1 & 1 & -1 & 1 & 0 & i & -i & 1 & -1 \\ 1 & -i & -1 & i & 0 & 1 & i & -i & -1 & 1 \end{pmatrix}. \tag{D.7}$$

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